Coinvariants of metaplectic representations on moduli of abelian varieties

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Motivation

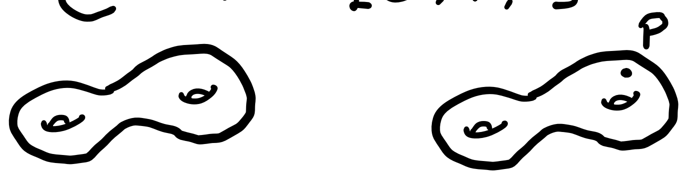
CFT $\Longrightarrow$

$\Uparrow$

a VOA $V \circlearrowleft$ Vir

quantum alg.

$$\begin{tikzcd}
\mathbb{V}(V) & \widehat{\mathbb{V}}(V) \arrow[l] & V/\mathcal{L}_{C\setminus P}(V)\cdot V \arrow[l, mapsto] \arrow[dl, bend left] \\
\mathcal{M}_g \arrow[u, leftarrow] & \widehat{\mathcal{M}}_g \arrow[l] \arrow[u, leftarrow] & \\
C \arrow[u, phantom, "\in" {rotate=90}] & {[C, P, t]} \arrow[l, mapsto] \arrow[u, phantom, "\in" {rotate=90}] &
\end{tikzcd}$$



$\widehat{\mathbb{V}}(V)$ is the v. bdle of coinvariants / $\widehat{\mathcal{M}}_g$

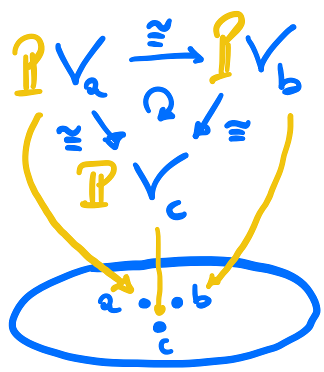
and carries a (projective) action of $\boxed{\mathcal{T}_{\widehat{\mathcal{M}}_g}}$

tangent sheaf

$\Vert$ w. equivariance w.r.t. gp. of change of t

[$\Rightarrow$ $\mathbb{V}(V)$ has a (twisted) D-module structure]

i.e., a way to identify (the proj. of) infinitesimally nearby fibers independently of the path connecting the base pts.



* Need a description of $\tau_{\hat{\mathcal{M}}_g}$

The Witt algebra $\quad \mathrm{Witt} := \mathbb{C}((t))\partial_t \ni \sum_{p \geq p_0} a_p L_p$

$$\text{top. gen. by } L_p := -t^{p+1}\partial_t \quad \text{for } p \in \mathbb{Z}$$

$$[L_p, L_q] = (p-q)L_{p+q}$$

$\operatorname{Spec} \mathbb{C}((t))$

!!

This is the Lie alg. of v. fields on the punctured disc

$\equiv$ derivations of the ring $\mathbb{C}((t))$

The Virasoro alg. $\quad 0 \to \mathbb{C}\mathbb{1} \to \mathrm{Vir} \to \mathrm{Witt} \to 0$

$$[\mathbb{1}, L_p] = 0 \qquad \forall p \in \mathbb{Z}$$

$$[L_p, L_q] = (p-q)L_{p+q} + \frac{1}{12}(p^3-p)\,\delta_{p+q,0}\,\mathbb{1}$$

$\delta_{p+q,0} = \begin{cases} 1 & p+q=0 \\ 0 & \text{oth.} \end{cases}$

Thm (ADKP, BS) Witt acts transitively on $\hat{\mathcal{M}}_g$:

1988

$$\mathrm{Witt} \longrightarrow\!\!\!\!\!\rightarrow_{Lie} \tau_{\hat{\mathcal{M}}_g}\big|_{[C,P,t]} \quad \text{for } [C,P,t] \in \hat{\mathcal{M}}_g.$$

Moreover $\quad \begin{matrix} \mathrm{Vir} \\ \downarrow \\ \mathrm{Witt} \end{matrix}$ acts transitively on $\begin{matrix} \Lambda \\ \downarrow \\ \hat{\mathcal{M}}_g \end{matrix}$ = Hodge line bdle

$$\Lambda|_{[C,P,t]} \cong \Lambda^g H^0(C,\omega_C)$$

$\mathbb{1} \in \mathrm{Vir}$ acts as mult. by 2 on fibers of $\Lambda \to \hat{\mathcal{M}}_g$

Ruggiero Torelli 1913

One has an inclusion

$$\begin{array}{ccc} \mathcal{M}_g & \hookrightarrow & \mathcal{A}_g \\ C & \longmapsto & (J(C), \Theta) \end{array}$$

$$J(C) = H^0(C, \Omega_C)^* / H_1(C, \mathbb{Z}) = \mathbb{C}^g / \Lambda$$

moduli space of deg 0 l. bdles on C

Arbarello – De Concini 1991

$\exists$ an ∞-dim'l analytic manifold $\hat{\mathcal{A}}_g$ parameterizing extended ab. var.

$=$ (extension of ab. var. by $t\mathbb{C}[[t]]$)$/\sim$

and a comm. diagram

$$\begin{tikzcd} \hat{\mathcal{M}}_g \arrow[r, hook, "\widehat{\text{Torelli}}"] \arrow[d] & \hat{\mathcal{A}}_g \arrow[d] & \hat{\mathcal{X}}_g \arrow[l] \arrow[d] \\ \mathcal{M}_g \arrow[r, "\text{Torelli}"] & \mathcal{A}_g & \mathcal{X}_g \arrow[l] \end{tikzcd}$$

univ. family

Q. What about the infinitesimal picture?

Shed the ring structure of $\mathbb{C}((t))$!

consider the bilinear + alternating form

$$\langle f, g \rangle := -\operatorname{Res}_{t=0} f\, dg \qquad f, g \in \mathbb{C}((t))$$

⇒ $\boxed{\text{Heisenberg alg } H} = \left(\mathbb{C}((t)), \langle , \rangle\right)$

2-nilpotent ⇒ Jacobi holds trivially

⚠ $\langle f, g\rangle = 0 \ \forall g \iff f = \text{const}$

⇒ $\langle , \rangle$ is non-degenerate on $\boxed{H' = \mathbb{C}((t))\big/\mathbb{C}\cdot t^0}$ symplectic v. sp.

⇒ $\mathfrak{sp}(H') := \left\{X \in \mathfrak{gl}(H') : \langle Xa, b\rangle + \langle a, Xb\rangle = 0 \ \ \forall a, b \in H'\right\}$

symplectic alg.

$$\| $$

$$\tilde{S}^2(H')$$

$ab : H' \longrightarrow H'$

$h \longmapsto \langle a, h\rangle b + \langle b, h\rangle a$

⚠ $\text{Witt} \xrightarrow{\text{oscillator}} \mathfrak{sp}(H')$

$f\partial_t \longmapsto f\partial_t$

$L_p \longmapsto \frac{1}{2}\sum_{i\in\mathbb{Z}} t^i \circ t^{-i+p}$

Arbarello – De Concini

1991

$\mathfrak{sp}(H') \longrightarrow T_{\hat{\mathcal{A}}_g}\big|_a \quad \forall a \in \hat{\mathcal{A}}_g$

$\mathfrak{sp}(H') \ltimes H' \longrightarrow T_{\hat{\mathcal{X}}_g}\big|_x \quad \forall x \in \hat{\mathcal{X}}_g$

Corollary

(ADKP, BS, AD, T.)

$$\begin{tikzcd}
\mathrm{Vir} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{mp}(H') \arrow[r, shift left] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathcal{U}}_2(H) \arrow[l, shift left] \arrow[d] \\
\mathrm{Witt} \arrow[r] & \mathfrak{sp}(H') \arrow[r, shift left] & \mathfrak{sp}(H') \ltimes H' \arrow[l, shift left] \\
\Lambda \arrow[r] \arrow[d] & \Lambda \arrow[d] & H \arrow[d] \\
\hat{\mathcal{M}}_g \arrow[r] & \hat{\mathcal{A}}_g & \hat{\mathcal{X}}_g \arrow[l]
\end{tikzcd}$$

acts transitively on

$1 \in \mathfrak{mp}(H')$ acts as mult. by 2 on fibers $\Lambda \downarrow \hat{\mathcal{M}}_g$

$1 \in \tilde{\mathcal{U}}_2(H)$ acts as mult. by -1 on fibers $H \downarrow \hat{\mathcal{X}}_g$

Def. $\left.\begin{array}{l}\text{Extended (p.p.) ab. var.} \\ a \in \hat{\mathcal{A}}_g\end{array}\right.$ $+$ $\left.\begin{array}{c} V \\ \circlearrowright \\ \mathfrak{mp}(H')\end{array}\right] \Rightarrow$

① $\mathfrak{sp}_a(H') \subseteq \mathfrak{mp}(H')$

$\mathfrak{sp}_a(H') = \operatorname{Ker}\left(\mathfrak{sp}(H') \longrightarrow T_a(\hat{\mathcal{A}}_g)\right)$

② a v. sp. of coinvariants

$$V \big/ \mathfrak{sp}_a(H') \cdot V$$

Thm (T., 2023)

① The spaces of coinv. give rise to a q.-coh. sheaf $\hat{\mathbb{V}}(V)$ on $\hat{\mathcal{A}}_g$ carrying a proj. action of $\mathcal{T}_{\hat{\mathcal{A}}_g}$ (hence a (twisted) D-mod str.)

② Under some natural assumptions, $\hat{\mathbb{V}}(V)$ descends (to a (twisted) D-mod) on $\mathcal{A}_g$.

③ Similarly, one obtains twisted D-mod on $\hat{\mathcal{X}}_g$ and $\mathcal{X}_g$.

* $\mathrm{rk}\left(\mathbb{V}_{\mathcal{A}_g}\right) = \infty$ in general!

<u>cfr.</u>: $\mathrm{rk}\left(\mathbb{V}_{\mathcal{M}_g}\right) < \infty$ in several cases!

⚠ $\mathfrak{sp}_2(H')$ is the minimal Lie subalg. of $\mathfrak{mp}(H')$ s.t. $\mathfrak{mp}(H') \curvearrowright V$ factors to $\mathcal{T}_{\hat{\mathcal{A}}_g} \curvearrowright \mathbb{P}\left(V/_{\mathfrak{sp}_2(H')\cdot V}\right)$

$$\begin{tikzcd}
\mathcal{T}(C\setminus P) \arrow[r] \arrow[d, hook] & \mathfrak{sp}_2(H') \arrow[d, hook] \\
\mathcal{L}_{C,P}(V) \arrow[r] & \boxed{?}
\end{tikzcd}$$

<u>Pb 1</u> Enlarge $\mathfrak{sp}_2(H')$ so that $\mathrm{rk}\,\mathbb{V}_{\hat{\mathcal{A}}_g} < \infty$.

<u>Pb 2</u> Extend $\mathbb{V}$ to $\mathcal{A}_g^{\mathrm{2Vor}}$. Factorization?