# Geometry of VOAs on moduli of curves

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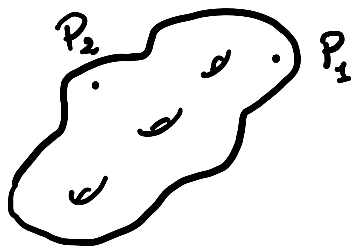
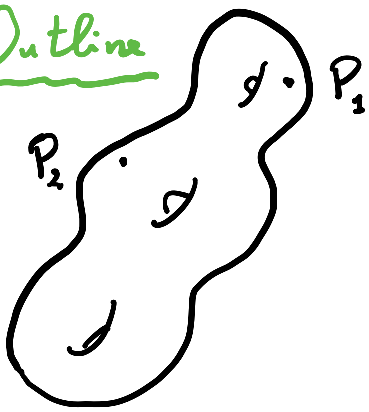
**Goal** Construct new vector bundles /

$\mathcal{M}_{g,n} \ni$ 


**Strategy** Apply Conformal Field Theory

**Outline**




$[C, P, t]$ + $\left[\begin{array}{l}\text{vertex operator} \\ \text{alg. } V \\ \text{and} \\ M=(M^1,\dots,M^n) \\ \text{s.t. } M^i \text{ is a} \\ V\text{-mod}\end{array}\right] \Rightarrow$

① Lie alg. $\mathcal{L}_{C\setminus P}(V)$

② $\mathbb{V}(V;M)_{[C,P,t]} = \bigotimes_{i=1}^{n} M^i \Big/ \mathcal{L}_{C\setminus P}(V)\cdot \bigotimes_{i=1}^{n} M^i$

This is the vector space of coinvariants

Its dual is the v. sp. of conformal blocks

* Extend this in families of curves
* get rid of the coordinates $t$

$\Rightarrow$

$$\begin{array}{c}\mathbb{V}(V;M)\\ \big\downarrow \text{ q.-coh. sheaf}\\ \mathcal{M}_{g,n}\end{array}$$

Over $\mathcal{M}_{g,n}$: Frenkel–Ben-Zvi
after Beilinson–Drinfeld

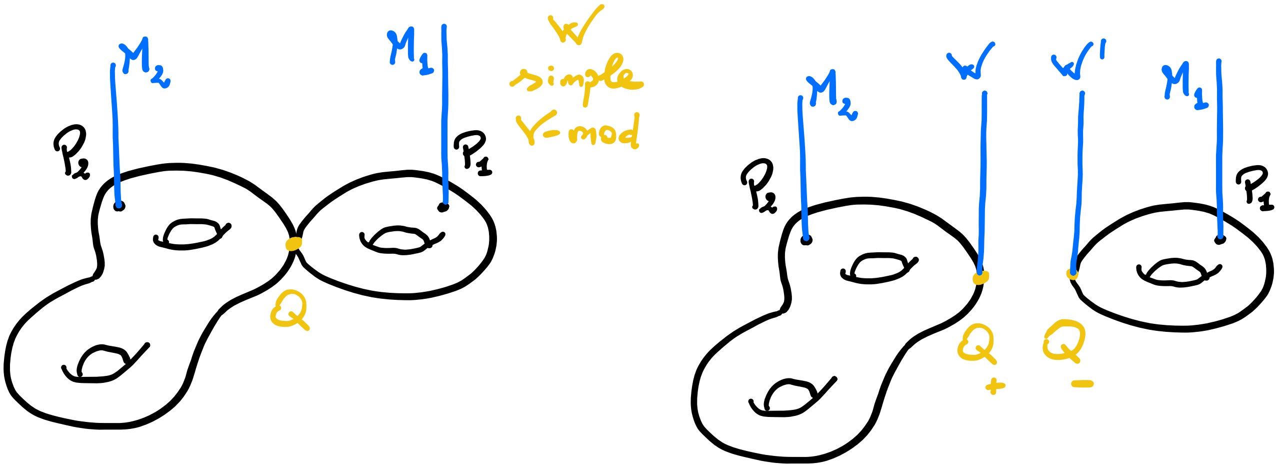
Conj. (Zhu, 1994; FBZ ~'00)

Under some natural assumptions on $V$:

① $\mathbb{V}(V;M)$ can be extended / $\overline{\mathcal{M}}_{g,n}$ and is a vector bundle;

② Factorization:

$$\mathbb{V}(V;M) \cong \bigoplus_{\substack{W \\ \text{simple} \\ V\text{-mod}}} \mathbb{V}(V;M^+,W)\otimes\mathbb{V}(V;M^-,W')$$




* For $g=0$: [Nagatomo–Tsuchiya]

VOAs can be induced from:

* affine Lie alg. [Tsuchiya-Ueno-Yamada, '89]
* Virasoro alg. [Beilinson-Feigin-Mazur, '91]
* pos. def. lattices [Borcherds '86]

Example: Even lattices [Frenkel-Lepowski-Meurman '88]

Input: a pos-def. even lattice

* a free abelian gp $L$ of finite rank
* a pos. def. bil. form $(\cdot,\cdot)$ on $L$ s.t.

$$(\alpha,\alpha)\in 2\mathbb{Z} \qquad \forall \alpha\in L$$

$\rightsquigarrow V_L$ even lattice VOA

* $$\{\text{simple } V_L\text{-mod}\}/\sim \;\xrightarrow{\cong}\; L'/L$$
$$V_{\lambda+L} \longmapsto \lambda+L$$

where $\underset{\text{dual lattice}}{L'} := \{\lambda\in L\otimes_{\mathbb{Z}}\mathbb{Q} : (\lambda,\mu)\in\mathbb{Z} \;\; \forall \mu\in L\}$

* contragredient $V_{\lambda+L}' = V_{-\lambda+L}$

Thm (Damiolini - Gibney - T.)

When $V$ is rational, $C_2$-cofinite, and $V_0 \cong \mathbb{C}$,

the Conj. holds.

Furthermore:

① $\mathbb{V}(V; M)$ is a v. bdle $/ \overline{\mathcal{M}}_{g,n}$

with a proj. flat log. connection

② Factorization holds

③ If $V = V'$ and simple, then $\left(\mathrm{ch}\left(\mathbb{V}(V; M)\right)\right)_{g,n}$ $\cap$ $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

is a semi-simple CohFT

For affine Lie alg: [TUY], [Tsuchimoto], [MOPPZ]

$V$ is $C_2$-cofinite: span$\{A_{(-2)}B : A, B \in V\}$ has finite codim in $V$

$V$ is rational: every finitely generated $V$-mod decomposes as a direct sum of simple $V$-mod's

A vertex operator alg. is: $\left(V, \mathbb{1}, \omega, Y(\cdot, t)\right)$

$$V = \bigoplus_{i \geq 0} V_i, \quad \mathbb{1} \in V_0, \quad \omega \in V_2, \quad Y(\cdot,t) : V \longrightarrow \mathrm{End}(V)[\![t, t^{-1}]\!]$$

$$A \longmapsto \underbrace{\sum_{i \in \mathbb{Z}} A_{(i)} t^{-i-1}}_{\text{vertex operators}}$$

+ axioms...

Idea the vertex operators endow $V$ with a weakly commutative alg. str.

* $\infty$'ly many products: $A *_i B := A_{(i)} B \quad \forall i \in \mathbb{Z}$
* unit $\mathbb{1}$ : $\mathbb{1}_{(-1)} = id_V$

  $\mathbb{1}_{(i)} = 0 \quad , i \neq -1$
* the vertex operators of $\omega$ + $\mathbb{1}_{(-1)}$

  $\Rightarrow Vir \curvearrowright V$ :

  $\omega_{(p+1)} \equiv L_p$

  $$[L_p, L_q] = (p-q) L_{p+q} + \frac{c}{12} \delta_{p+q=0} (p^3 - p)\, id_V$$

  central charge

⚠ Frenkel-Ben-Zvi: Regard $t$ as a formal coordinate at a pt $P \in C$!

$\rightsquigarrow V$ given $\quad \begin{array}{c}\mathcal{V}_C \\ \downarrow \\ C\end{array}$ sheaf

Properties:

$*\quad \bigoplus_{k \geq 0} \mathcal{V}_{\leq k} / \mathcal{V}_{\leq k-1} =: \boxed{\mathrm{gr}_\bullet \mathcal{V}_C = \bigoplus_{k \geq 0} \left(\omega_C^{\otimes -k}\right)^{\oplus \dim V_k}}$

$*\quad$ a flat log. connection $\nabla : \mathcal{V}_C \longrightarrow \mathcal{V}_C \otimes \omega_C$

locally on smooth open $U \subseteq C$ : $\quad U \xrightarrow{\text{ét.}} \mathbb{A}^1$

$**\quad \mathcal{V}_U \simeq_t V \times U$

$**\quad \nabla = L_{-1} \otimes id_U + id_V \otimes \partial_t$

$\rightsquigarrow$ get a sheaf of Lie alg. on $C$ :

$$\mathcal{L}_{U \subseteq C}(V) := H^0(U, \mathcal{V}_C \otimes \omega_C / \nabla)$$

Idea ① $\mathcal{L}_{C \setminus P}(V)$ = Lie alg. used in def. of coinvariants

② $\mathcal{L}_{\times}(V) \xrightarrow{\text{Lie}} \mathrm{End}(V)$

Spec $\mathbb{C}((t)) =: \times \quad A_{[i]} \longmapsto A_{(i)}$ vertex op.

$$\mathrm{Vir} \xrightarrow[t]{} \mathcal{L}_{\times}(V) \curvearrowright M$$

Virasoro $\qquad$ ancillary $\qquad$ $V$-mod

$\rightsquigarrow$ $Vir \curvearrowright M^i$ induces a proj. flat log. connection

Cor1(DGT) Assume $M^i$ is simple, $\forall i = 1, \dots, n$.

$$ch\left(\mathbb{V}(V;M)\big|_{\overline{\mathcal{M}}_{g,n}}\right) = \text{rank}\, \mathbb{V}(V;M) \cdot \exp\left(\frac{c}{2}\lambda + \sum_{i=1}^{n} a_i \psi_i\right)$$

$\cap$
$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

where
* $c$ = central charge of $V \in \mathbb{Q}$
* $a_i$ = conformal dim of $M^i \in \mathbb{Q}$
* $\lambda = c_1$(Hodge line bdle)
* $\psi_i = c_1$(cotg line bdle at $i$-th pt)

Cor2(DGT) The formula for $ch(\mathbb{V}(V;M))$ in aff. Lie alg. case computed by $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ [Marian-Oprea-Pandharipande-Pixton-Zvonkine] extends to the case of VOA of CohFT-type.

## Question (Pandharipande)

Find an alternative way to compute $ch(\mathbb{V}(V; M))$