Incidence Varieties of Algebraic Curves and Canonical Divisors

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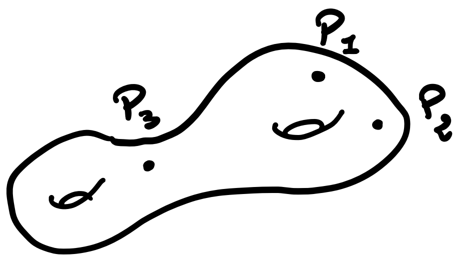
Goal Study the enumerative geometry of loci of curves with canonical div's

Start $\mathcal{M}_{g,n} \ni$ 

$= [C, P]$

Proj. Hodge

$$\begin{tikzcd}
\mathbb{P}\mathbb{E}_{g,n} \arrow[d] & \mathbb{P}H^0(C,\omega_C) \arrow[l, hook'] \arrow[d] \\
\overline{\mathcal{M}}_{g,n} \ni & {[C,P]}
\end{tikzcd}$$

proj. v. sp. of diff. on C

$\in$

$[C,P,\mu]$

Given $m=(m_1,\dots,m_n)$ s.t. $m_i \geq 0$, define

$$\overline{\mathbb{H}}_{g,m} := \overline{\left\{ [C,P,\mu] \in \mathbb{P}\mathbb{E}_{g,n} : \begin{array}{l} C \text{ smooth } + \\ \mu \text{ vanishes at } P_i \\ \text{with order } m_i,\ \forall i \end{array} \right\}}$$

$\boxed{\text{incidence varieties}}$

Context

$$\begin{tikzcd}
\mathbb{P}\mathbb{E}_{g,n} \arrow[d] & \overline{\mathbb{H}}_{g,m} \arrow[l, hook'] \arrow[d] \\
\overline{\mathcal{M}}_{g,n} & \overline{\mathcal{H}}_{g,m} \arrow[l, hook']
\end{tikzcd}$$

* Farkas - Pandharipande, '15
* Janda - Pandharipande - Pixton - Zvonkine, '15
* Schmitt, '16
* Bae - Holmes - Pandharipande - Schmitt - Schwarz, '20

Conclusion

There exists an algorithm to compute $[\overline{\mathcal{H}}_{g,m}]$, but no closed formula

Our Results

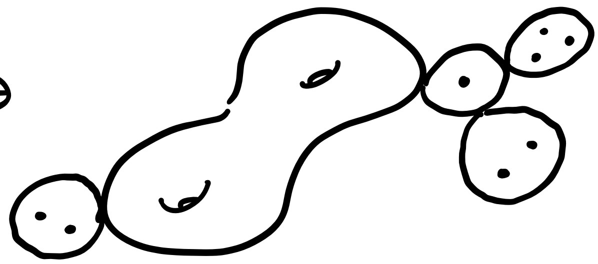
A closed formula for $[\overline{\mathbb{H}}_{g,m}]$ over $\mathcal{M}_{g,n}^{rt} \subset \overline{\mathcal{M}}_{g,n}$

* Bainbridge, Chen, Gendron, Grushevsky, Möller '16
* Sauvaget, '17

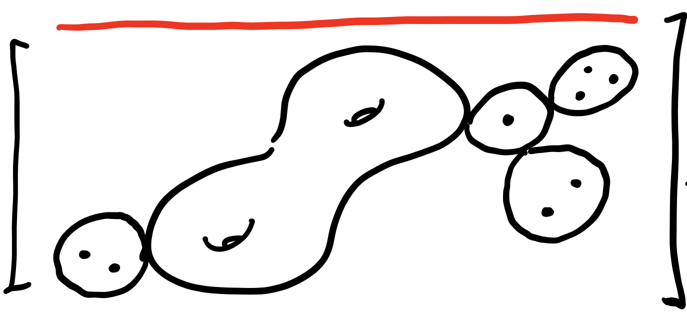
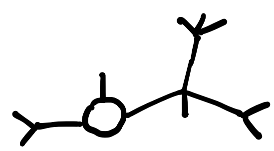
Fact

$$\overset{R^*}{A^*}(\mathbb{P}(\mathbb{E}_{g,n})) = \overset{R^*}{A^*}(\overline{\mathcal{M}}_{g,n})[\eta] \Big/ \sum_i (-\eta)^i c_{r-i}(\mathbb{E}_{g,n})$$

$$\eta := c_1\left(\mathcal{O}_{\mathbb{P}(\mathbb{E}_{g,n})}(-1)\right)$$

* $\mathcal{M}^{rt}_{g,n} \ni$ 

Boundary Strata of $\mathcal{M}^{rt}_{g,n} \xleftrightarrow{\text{bij}} G^{rt}_{g,n}$

 $\longleftrightarrow$  $= \Gamma$

For simplicity :

* $k = 1$

* reduce to $m = (1, \dots, 1)$

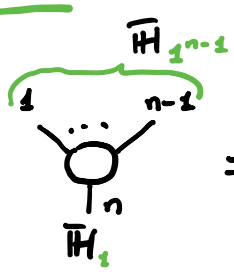
* $n=1$: $\quad [\overline{\mathbb{H}}_{g,1}] = \psi - \eta \qquad$ [Sauvaget]

* Proceed by recursion:

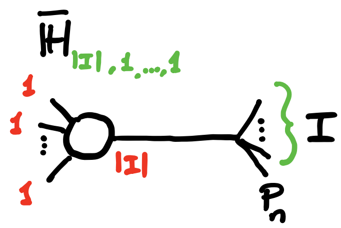
$$\begin{tikzcd}[column sep=small]
 & \underset{[C,P_1,\dots,P_n,\mu]}{\overset{\Psi}{\mathbb{P}(E_{g,n})}} \arrow[dl, "\pi_n"'] \arrow[dr, "\rho_n"] & \\
\underset{[C,P_1,\dots,P_{n-1},\mu]}{\overset{\Psi}{\mathbb{P}(E_{g,n-1})}} & & \underset{[C,P_n,\mu]}{\overset{\Psi}{\mathbb{P}(E_{g,1})}}
\end{tikzcd}$$

$$\pi_n^*\left[\overline{\mathbb{H}}_{g,1^{n-1}}\right]\cdot \rho_n^*\left[\overline{\mathbb{H}}_{g,1}\right] = \left[\overline{\mathbb{H}}_{g,1^n}\right] + \cdots$$

Thm (Sauvaget; Gheorghita-T.) For $n \geq 2$:



$= \left[\overline{\mathbb{H}}_{1^n}\right] + \sum_{I \subseteq P \setminus P_n} |I|$



holds in $A^n\left(\mathbb{P}\left(E^*_{g,n}\big|_{\mathcal{M}^{ct}_{g,n}}\right)\right)$.

* To solve the recursion, expand on [Cavalieri-T.]

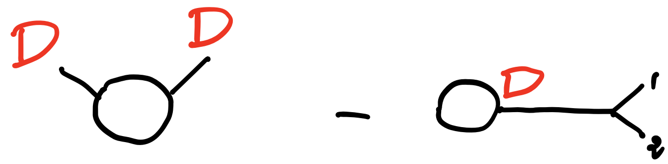
Ex

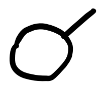
$D := \omega - \eta$

where

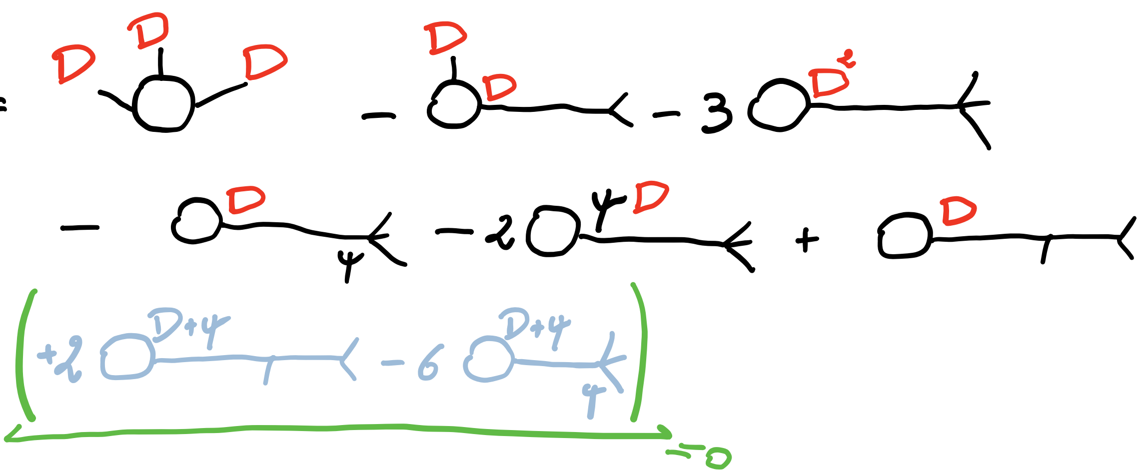
$\omega_i := \rho_i^* \psi$

$\overline{\mathbb{H}}_{(1,1)} =$

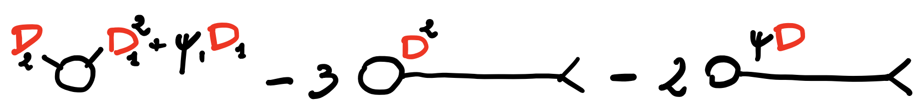


$\overline{\mathbb{H}}_{(2)} =$  $(D^2 + \psi D)$

$\overline{\mathbb{H}}_{(1,1,1)} =$

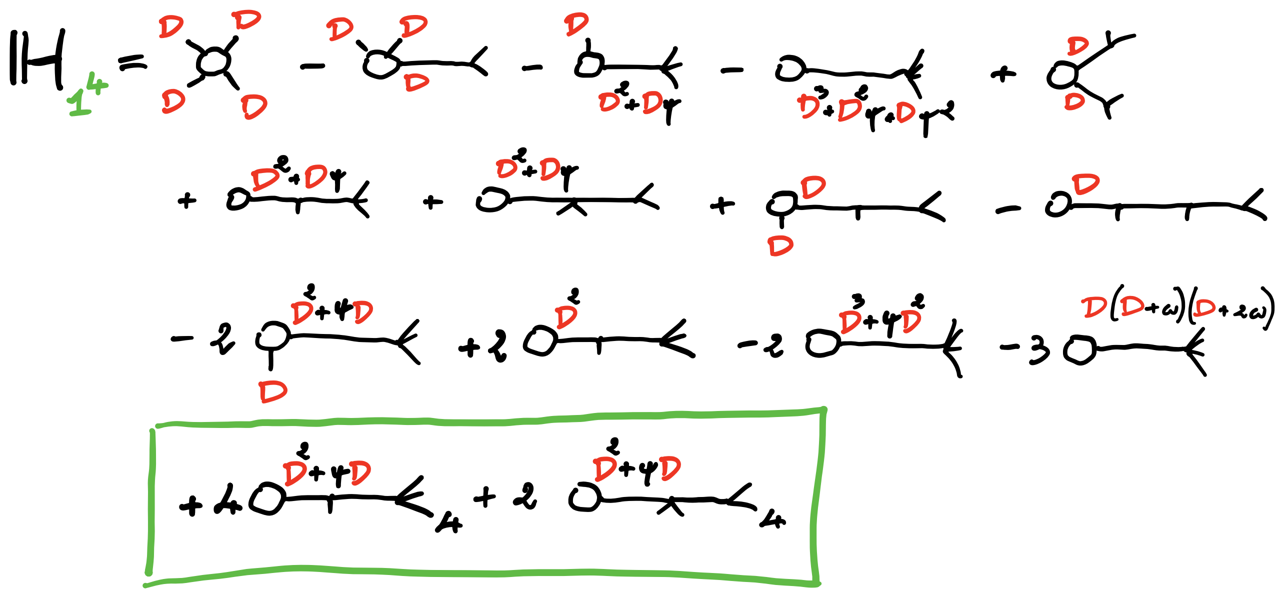


$\overline{\mathbb{H}}_{(2,1)} =$

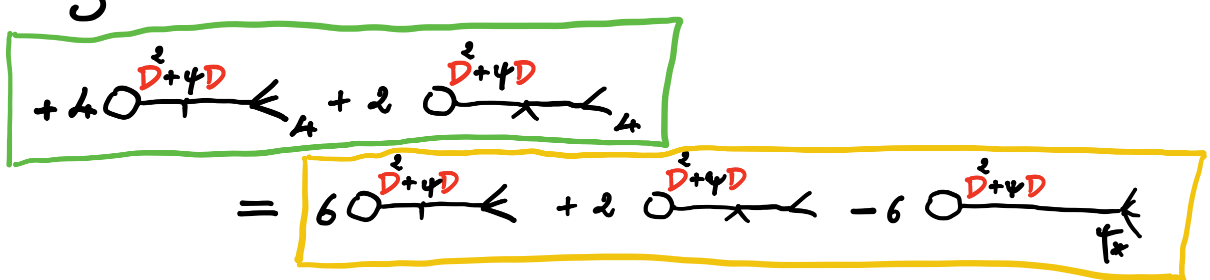


$$\overline{\mathbb{H}}_{(3)} = (D + 2\psi)(D + \psi) D$$

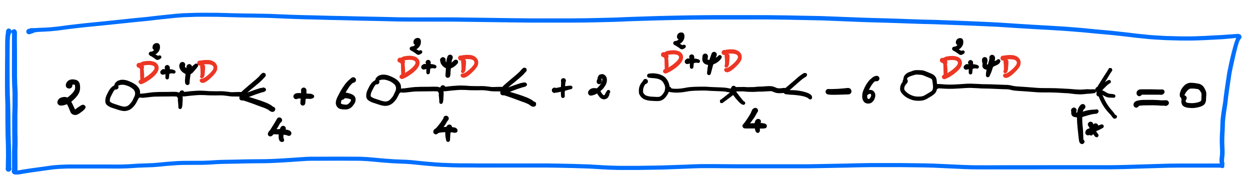
$$= D^3 + 3\psi D^2 + 2\psi^2 D$$



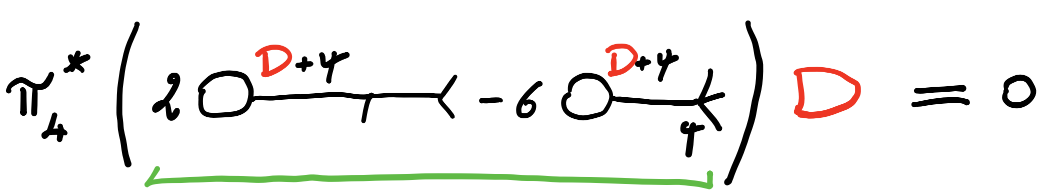
To symmetrize, use:



This identity is equivalent to:

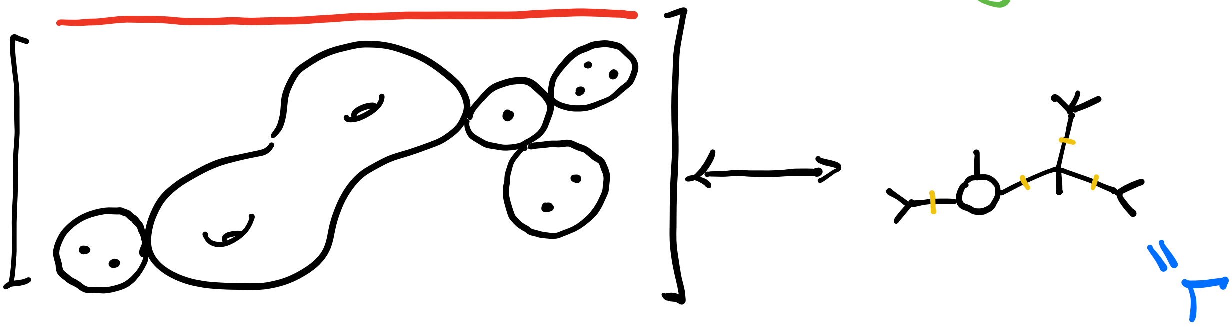


In turn, this is



ghost term for $n = 3$

Boundary Strata of $\overline{\mathcal{M}}^{rt}_{g,n}$ $\xleftrightarrow{\text{bij}}$ $G^{rt}_{g,n}$



Decorations $\quad \psi = \prod_{h \in H(\Gamma)} \psi_h^{d_h}$

For simplicity: $\; m = (1, \dots, 1)$

The graph Formula

$$F_{g,n} := \sum_{\Gamma, \psi} (-1)^{|E(\Gamma)|} \underbrace{c_{\Gamma,\psi}}_{\in \mathbb{Z}_{>0}} T_{\Gamma,\psi} \in R^n(\mathbb{P}E_{g,n})$$

Features

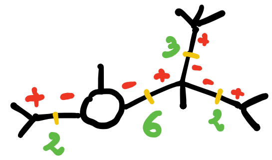
* a lin. comb. of taut. classes $T_{\Gamma,\psi}$ of deg n
* contains ghost terms which sum to zero!
* no decorations on vertices!

Thm (Gheorghita–T.) For $g \geq 2$,

$$[\overline{\mathbb{H}}_{g,n}] = F_{g,n} \quad \in A^n\left(\mathbb{P}E_{g,n}\Big|_{\mathcal{M}_{g,n}^{rt}}\right)$$

* To define $c_{\Gamma,\psi}$:

Capacity ℓ_{h^+} :



rooted tree

Set of dec. half-edges

$$H(\Gamma,\psi) := \left\{ (h,a) : \begin{array}{l} h \in H^+(\Gamma),\ 0 \leq a \leq d_h \\ h \in H^-(\Gamma),\ 1 \leq a \leq d_h \end{array} \right\}$$

* $E(\Gamma) \xleftrightarrow{\text{bij}} \{(h^+,0)\}$
* $\deg \psi = |\{(h,a)\}|$

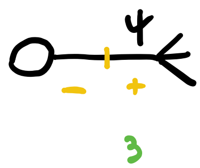
$$\Rightarrow |H(\Gamma,\psi)| = \deg\psi + |E(\Gamma)|$$

Weightings

$$W_{\Gamma,\psi} := \left\{ w : H(\Gamma,\psi) \longrightarrow \mathbb{N} : \begin{array}{c} \ell_{h^+} > w(h^+,0) \geq \cdots \geq w(h^+, d_{h^+}) \\ \text{for all heads } h^+ \\ 1 \leq w(h^-,1) < \cdots < w(h^-, d_{h^-}) < w(h^{-+},0) \\ \text{for all tails } h^- \end{array} \right\}$$

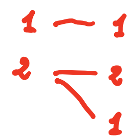
Def $$c_{\Gamma,\psi} := \sum_{w\in W_{\Gamma,\psi}} \prod_{(h,a)} w(h,a)$$

Ex $(\Gamma,\psi) =$

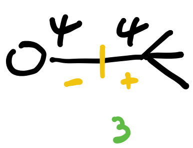


$H(\Gamma,\psi)=\{(h^+,0),(h^+,1)\}$

$$\boxed{c_{\Gamma,\psi} = 7}$$

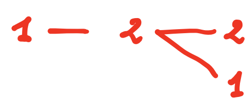


Ex $(\Gamma,\psi) =$

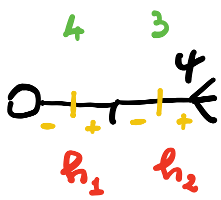


$H(\Gamma,\psi)=\{(h^-,1),(h^+,0),(h^+,1)\}$

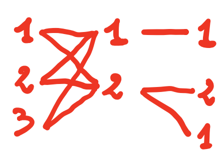
$$\boxed{c_{\Gamma,\psi} = 6}$$



Ex $(\Gamma,\psi) =$



$H(\Gamma,\psi)=\{(h_1^+,0),(h_2^+,0),(h_2^+,1)\}$



$$\boxed{c_{\Gamma,\psi} = 6\cdot 7 = 42}$$

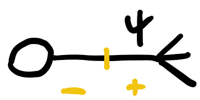
* To define $T_{\Gamma,\psi}$:

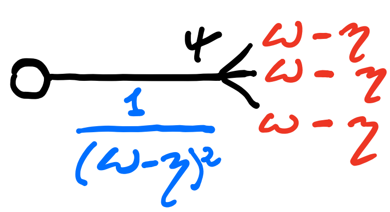
$$\begin{tikzcd} \mathbb{P}\mathbb{E}|_{\mathcal{M}_\Gamma} \arrow[r, "\xi_\Gamma"] \arrow[d] & \mathbb{P}\mathbb{E}_{g,n} \arrow[d] \\ \mathcal{M}_\Gamma := \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \arrow[r] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

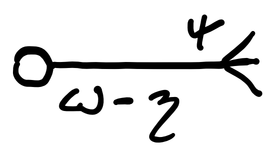
$$T_{\Gamma,\psi} := \xi_{\Gamma *}\left(\left(\prod_{i=1}^{n} (\omega_i - z) \right) \psi \, \beta_{\Gamma,\psi}^{-1} \right)$$

where: * $\omega_i := \rho_i^* \psi$, $\rho_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,1}$
$[C,P] \longmapsto [C,P_i]$

* $\beta_{\Gamma,\psi} := \prod_{(h^+,0)} (\omega_{h^+} - z) \prod_{\substack{(h,a) \\ a\neq 0}} (\omega_h - z)$

Ex $(\Gamma,\psi) =$ 

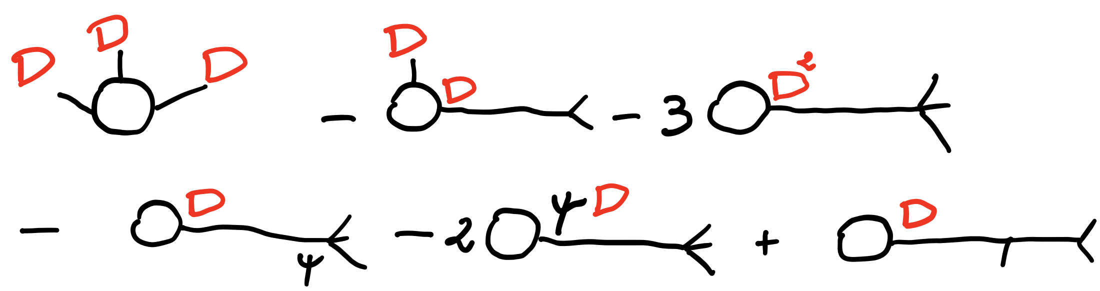
$T_{\Gamma,\psi} =$ 

$=$ 

Cor. (Gheorghita–T.) For $g \geq 2$ and $n > 2g-2$,

$$F_{g,n} = 0 \quad \text{in } R^n\left(\mathbb{P}\mathbb{E}_{g,n}\Big|_{\mathcal{M}^{rt}_{g,n}}\right)$$

Ex $g=2$, $n=3$:



$$= 0 \quad \text{in } R^3\left(\mathbb{P}\mathbb{E}_{2,3}\Big|_{\mathcal{M}^{rt}_{2,3}}\right)$$

$\Rightarrow$ $\left(\text{coeff. of } \eta^a\right) = 0$ in $R^{3-a}\left(\mathcal{M}^{rt}_{2,3}\right)$

* $a = 1$ $\Rightarrow$ Belorousski–Pandharipande relation in $R^2\left(\mathcal{M}^{rt}_{2,3}\right)$

Q $\quad F_{g,n} = 0 \overset{?}{\Longrightarrow}$ new relation

Q $\quad \{F_{g,n} = 0\} \overset{?}{\subseteq}$ Faber-Zagier-Pixton