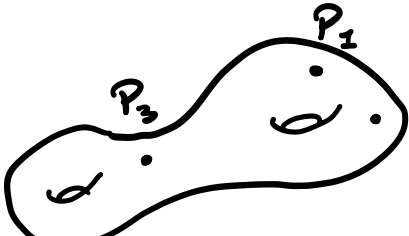


Incidence Varieties of Algebraic Curves and Canonical Divisors

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Goal Study the enumerative geometry of loci of curves with canonical divisors

Start  $\mathcal{M}_{g,n} \ni$    $= [C, P]$

Proj. Hodge  $\mathbb{P}E_{g,n} \xrightarrow{\sim} \mathbb{P}H^0(C, \omega_C)$  proj. v. sp.  
of diff.  
on C

$\downarrow$   $\downarrow$   $\downarrow$   
 $\mathcal{M}_{g,n} \ni [C, P] \quad [C, P, \mu]$

Given  $m = (m_1, \dots, m_n)$  s.t.  $m_i \geq 0$ , define

$\mathbb{H}_{g,m} := \{ [C, P, \mu] \in \mathbb{P}E_{g,n} : C \text{ smooth + } \mu \text{ vanishes at } P_i \text{ with order } m_i, \forall i \}$

incidence varieties

## Context

$$\begin{array}{ccc} \mathbb{P}E_{g,n} & \longleftrightarrow & \overline{H}_{g,m} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \longleftrightarrow & \overline{\mathcal{H}}_{g,m} \end{array}$$

- \* Farkas - Pandharipande, '15
- \* Janda - Pandharipande - Pixton - Zvonkine, '15
- \* Schmitt, '16
- \* Bae - Holmes - Pandharipande - Schmitt - Schwarz, '20

Conclusion There exists an algorithm to compute  $[\overline{\mathcal{H}}_{g,m}]$ . but no closed formula

Our Results A closed formula for

$$[\overline{H}_{g,m}] \text{ over } \mathcal{M}_{g,n}^{\text{zt}} \subset \overline{\mathcal{M}}_{g,n}$$

- \* Bainbridge, Chen, Gendron, Gianshchik, Höller, '16
- \* Sauvaget, '17

Fact

$\mathbb{R}^*$

$\mathbb{R}^*$

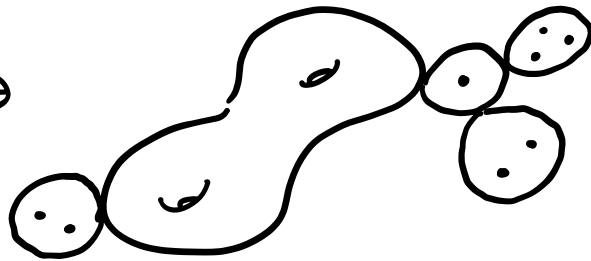
$$\eta := c_2 \left( \mathcal{O}_{\mathbb{P}(E_{g,n})}(-1) \right)$$

$$A^*(\mathbb{P}(E_{g,n})) = A^*(\bar{\mathcal{M}}_{g,n})[\eta]$$

$$\bigg/ \sum_i (-\eta)^i c_{2-i}(E_{g,n})$$

\*  $\mathcal{M}_{g,n}^{zt}$

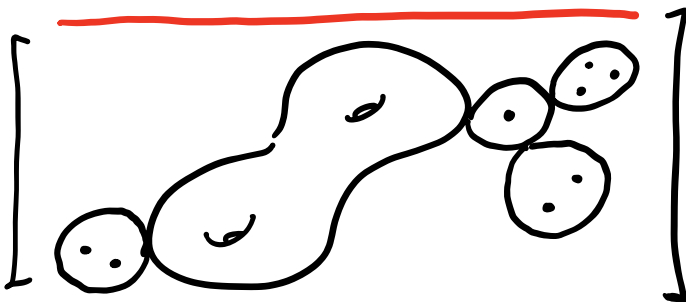
$\ni$



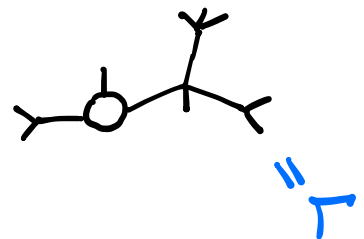
Boundary Strata of  $\mathcal{M}_{g,n}^{zt}$

$\xleftrightarrow{\text{bij}}$

$\mathcal{G}_{g,n}^{zt}$



$\longleftrightarrow$



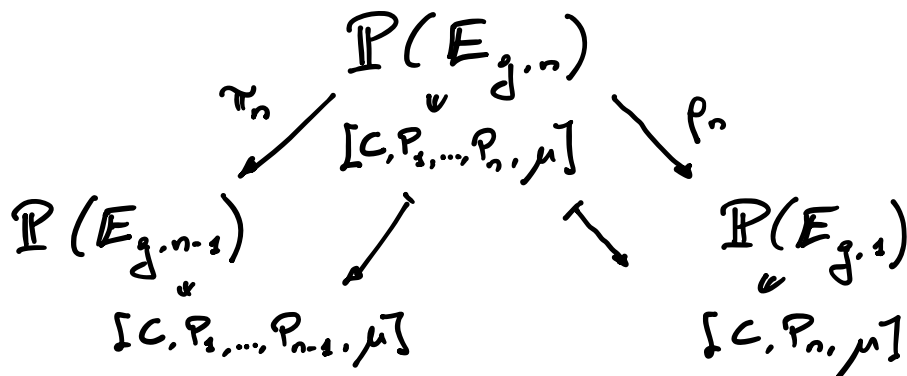
For simplicity :

\*  $g = 1$

\* reduce to  $m = (1, \dots, 1)$

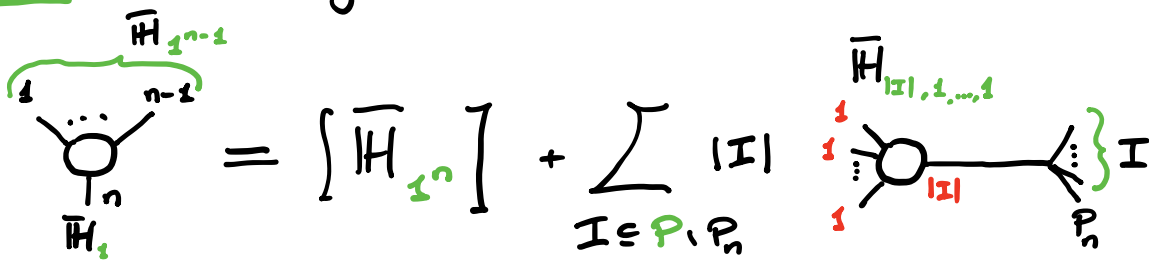
\*  $n=1$ :  $[\overline{H}_{j,1}] = \psi - \eta$  [Sawagot]

\* Proceed by recursion:



$$\pi_n^* [\overline{H}_{j,1}^{n-1}] \cdot \rho_n^* [\overline{H}_{j,1}] = [\overline{H}_{j,1}^n] + \dots$$

Lemma (Sawagot; Gheorghita-T.) For  $n \geq 2$ :



holds in  $A^n (P(E_{j,n}^k / \mathcal{H}_{j,n}^{n-2}))$ .

\* To solve the recursion, expand on [Cavalieri-1.]

$E_*$

$$\bar{H}_{(1)} = \begin{array}{c} D \\ \diagdown \\ \bigcirc \\ \diagup \\ D \end{array} - \begin{array}{c} \bigcirc \\ \diagup \\ D \\ \diagdown \\ \begin{array}{l} 1' \\ 2 \end{array} \end{array}$$

$$D := \omega - \gamma$$

where

$$\omega_i := \beta_i^* \psi$$

$$\bar{H}_{(2)} = \bigcirc (D^2 + \psi D)$$

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$$\begin{aligned} \bar{H}_{(1,1)} = & \begin{array}{c} D \quad D \\ \diagdown \quad \diagup \\ \bigcirc \\ \diagup \quad \diagdown \\ D \quad D \end{array} - \begin{array}{c} D \\ \diagdown \\ \bigcirc \\ \diagup \\ D \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} - 3 \begin{array}{c} \bigcirc \\ \diagup \\ D^2 \\ \diagdown \\ \diagup \end{array} \\ & - \begin{array}{c} \bigcirc \\ \diagup \\ D \\ \diagdown \\ \psi \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} - 2 \begin{array}{c} \bigcirc \\ \diagup \\ \psi D \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} + \begin{array}{c} \bigcirc \\ \diagup \\ D \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} \\ & \left( +2 \begin{array}{c} \bigcirc \\ \diagup \\ D+\psi \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} - 6 \begin{array}{c} \bigcirc \\ \diagup \\ D+\psi \\ \diagdown \\ \psi \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} \right) \rightarrow 0 \end{aligned}$$

$$\bar{H}_{(2,1)} = \begin{array}{c} D \\ \diagdown \\ \bigcirc \\ \diagup \\ D \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} + \psi \begin{array}{c} \bigcirc \\ \diagup \\ D^2 \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} - 3 \begin{array}{c} \bigcirc \\ \diagup \\ D^2 \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array} - 2 \begin{array}{c} \bigcirc \\ \diagup \\ \psi D \\ \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array}$$

$$\begin{aligned} \bar{H}_{(3)} &= (D + 2\psi)(D + \psi)D \\ &= D^3 + 3\psi D^2 + 2\psi^2 D \end{aligned}$$

$$\begin{aligned}
\mathbb{H}_{1^+} = & \text{Diagram 1} - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} + \text{Diagram 5} \\
& + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} - \text{Diagram 9} \\
& - 2 \text{Diagram 10} + 2 \text{Diagram 11} - 2 \text{Diagram 12} - 3 \text{Diagram 13} \\
& + 4 \text{Diagram 14} + 2 \text{Diagram 15}
\end{aligned}$$

(The diagrams are Feynman diagrams with various labels like  $D$ ,  $D^2+4D$ ,  $D^2$ ,  $D^3+4D^2$ , and  $D(D+2)(D+2)$ )

To symmetrize, use:

$$\begin{aligned}
& + 4 \text{Diagram 14} + 2 \text{Diagram 15} \\
& = 6 \text{Diagram 16} + 2 \text{Diagram 17} - 6 \text{Diagram 18}
\end{aligned}$$

(The diagrams are Feynman diagrams with labels  $D^2+4D$  and  $D^2$ )

This identity is equivalent to:

$$2 \text{Diagram 19} + 6 \text{Diagram 20} + 2 \text{Diagram 21} - 6 \text{Diagram 22} = 0$$

(The diagrams are Feynman diagrams with labels  $D^2+4D$  and  $D^2$ )

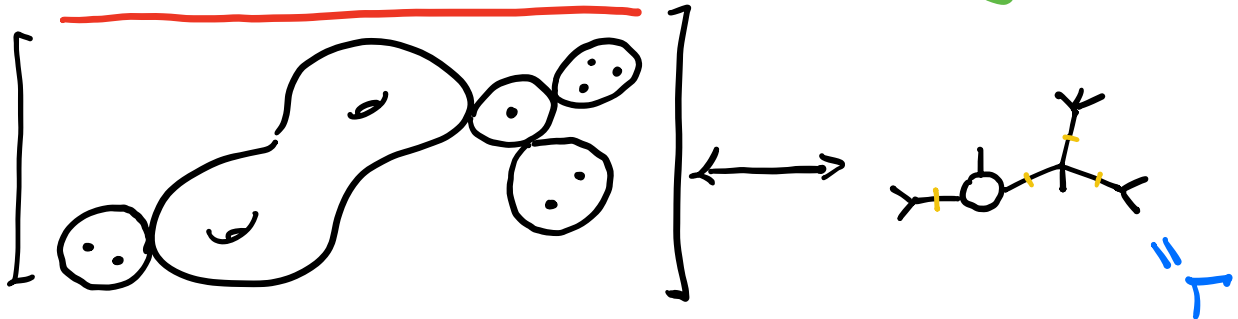
In turn, this is

$$\pi_4^* \left( 2 \text{Diagram 23} - 6 \text{Diagram 24} \right) D = 0$$

(The diagrams are Feynman diagrams with labels  $D+4$  and  $D^2$ )

ghost term for  $n=3$

Boundary Strata of  $\mathcal{M}_{g,n}^{zt}$   $\xleftrightarrow{\text{bij}}$   $G_{g,n}^{zt}$



Decorations  $\psi = \prod_{h \in H(\Gamma)} \psi_h^{d_h}$

For simplicity:  $m = (1, \dots, 1)$

The graph formula

$$F_{g,n} := \sum_{\Gamma, \psi} (-1)^{|E(\Gamma)|} \underbrace{c_{\Gamma, \psi}}_{\mathbb{Z}_{>0}} \tau_{\Gamma, \psi} \in \hat{\mathcal{R}}(\mathcal{P}E_{g,n})$$

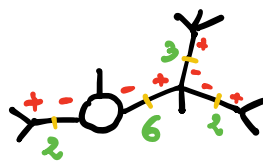
Features

- \* a lin. comb. of taut. classes  $\tau_{\Gamma, \psi}$  of deg  $n$
- \* contains **ghost terms** which sum to zero!
- \* no decorations on vertices!

Thm (Theor 7.1) For  $g \geq 2$ .

$$[\overline{H}_{g,n}] = F_{g,n} \in A^n \left( \mathbb{P}E_{g,n} \mid \mathcal{M}_{g,n}^{\pm} \right)$$

\* To define  $C_{\Gamma, \gamma}$ :

Capacity  $l_{h^+}$ :  rooted tree

Set of dec. half-edges

$$H(\Gamma, \gamma) := \left\{ (h, a) : \begin{array}{l} h \in H^+(\Gamma), 0 \leq a \leq d_h \\ h \in H^-(\Gamma), 1 \leq a \leq d_h \end{array} \right\}$$

$$\begin{array}{l} * E(\Gamma) \xleftrightarrow{\text{bij}} \{(h^+, 0)\} \\ * \deg \gamma = |\{(h, a)\}| \end{array} \Rightarrow |H(\Gamma, \gamma)| = \deg \gamma + |E(\Gamma)|$$

Weightings  $W_{\Gamma, \gamma} := \left\{ \begin{array}{l} w: H(\Gamma, \gamma) \rightarrow \mathbb{N} : \\ l_{h^+} > w(h^+, 0) \geq \dots \geq w(h^+, d_{h^+}) \\ \text{For all heads } h^+ \\ 1 \leq w(h^-, 1) < \dots < w(h^-, d_{h^-}) < w(h^-, 0) \\ \text{For all tails } h^- \end{array} \right\}$





\* To define  $T_{\Gamma, \psi}$ :

$$\begin{array}{ccc} \mathbb{P}E|_{\mathcal{H}_\Gamma} & \xrightarrow{\xi_\Gamma} & \mathbb{P}E_{g,n} \\ \downarrow & & \downarrow \\ \mathcal{H}_\Gamma := \prod_{\sigma \in V(\Gamma)} \overline{\mathcal{H}}_{g(\sigma), n(\sigma)} & \longrightarrow & \overline{\mathcal{H}}_{g,n} \end{array}$$

$$T_{\Gamma, \psi} := \xi_\Gamma * \left( \prod_{i=1}^n (\omega_i - \gamma) \right) \psi \beta_{\Gamma, \psi}^{-1}$$

where: \*  $\omega_i := \beta_i^* \psi$  .  $\beta_i: \overline{\mathcal{H}}_{g,n} \rightarrow \overline{\mathcal{H}}_{g,1}$   
 $[c, p] \mapsto [c, \beta_i]$

$$* \beta_{\Gamma, \psi} := \prod_{(h,0)} (\omega_h - \gamma) \prod_{(h, \frac{a}{0})} (\omega_h - \gamma)$$

Ex  $(\Gamma, \psi) = \text{O} \xrightarrow{\psi} \text{K}$

$$T_{\Gamma, \psi} = \text{O} \xrightarrow{\frac{1}{(\omega-\gamma)^2}} \text{K} \begin{matrix} \omega-\gamma \\ \omega-\gamma \\ \omega-\gamma \end{matrix} \psi = \text{O} \xrightarrow{\omega-\gamma} \text{K} \psi$$

Cor. (Gheorgheita-T.) For  $g \geq 2$  and  $n > 2g-2$ ,

$$F_{g,n} = 0 \quad \text{in } \mathbb{R}^n \left( \mathbb{P}E_{g,n} \Big|_{\mathcal{H}_{g,n}^{zt}} \right)$$

Ex  $g=2, n=3$ :

$$\begin{aligned} & \overset{D}{\circ} \overset{D}{\circ} \overset{D}{\circ} - \overset{D}{\circ} \overset{D}{\circ} \leftarrow -3 \overset{D^2}{\circ} \leftarrow \leftarrow \\ & - \overset{D}{\circ} \leftarrow \leftarrow \leftarrow -2 \overset{D}{\circ} \leftarrow \leftarrow \leftarrow + \overset{D}{\circ} \leftarrow \leftarrow \leftarrow \end{aligned}$$

$$= 0 \quad \text{in } \mathbb{R}^3 \left( \mathbb{P}E_{2,3} \Big|_{\mathcal{H}_{2,3}^{zt}} \right)$$

$$\Rightarrow (\text{coeff. of } z^a) = 0 \quad \text{in } \mathbb{R}^{3-a} \left( \mathcal{H}_{2,3}^{zt} \right)$$

\*  $a=1 \Rightarrow$  Belorousski-Pandharipande relation in  $\mathbb{R}^2 \left( \mathcal{H}_{2,3}^{zt} \right)$

$\mathbb{Q}$   $F_{g,n} = 0$   $\stackrel{?}{\Rightarrow}$  new relation

$\mathbb{Q}$   $\{F_{g,n} = 0\}$   $\stackrel{?}{\subseteq}$  Faber-Zagier-Pixton