Series Invariants for Plumbed 3-Manifolds

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Plumbings

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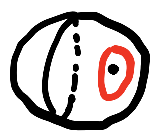
$$\overset{m}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{n}{\bullet} \longmapsto \left[\begin{array}{ccc} M(\overset{m}{\bullet}) & \sqcup & M(\overset{n}{\bullet}) \\ \downarrow S^1 & & \downarrow S^1 \end{array}\right] \Big/ \begin{array}{c} S^1 \times S^1 \cong S^1 \times S^1 \\ (x, y) \mapsto (y, x) \end{array}$$

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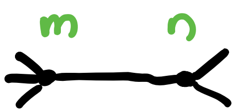
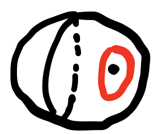


$$\longmapsto \left[M(\cdot^{m}) \sqcup M(\cdot^{n}) \right] \Big/ \begin{array}{c} S^1 \times S^1 \cong S^1 \times S^1 \\ (x, y) \mapsto (y, x) \end{array}$$

$M(\cdot^m) \downarrow S^1$, $M(\cdot^n) \downarrow S^1$



framing matrix $B = \begin{pmatrix} m & 1 \\ 1 & n \end{pmatrix}$

Idea $M = \partial X$ for a 4-mfd X and $B =$ int. form on $H_2(X, \mathbb{Z})$

Neumann

1981

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$\Longleftrightarrow$ Γ and Γ' are related by a sequence of moves

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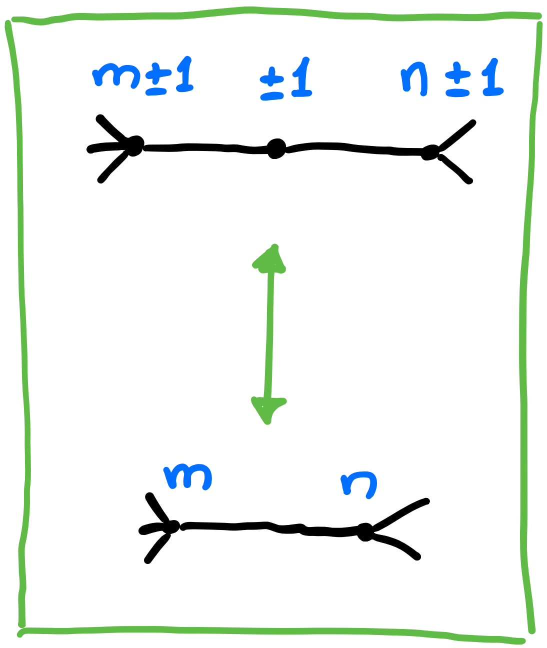
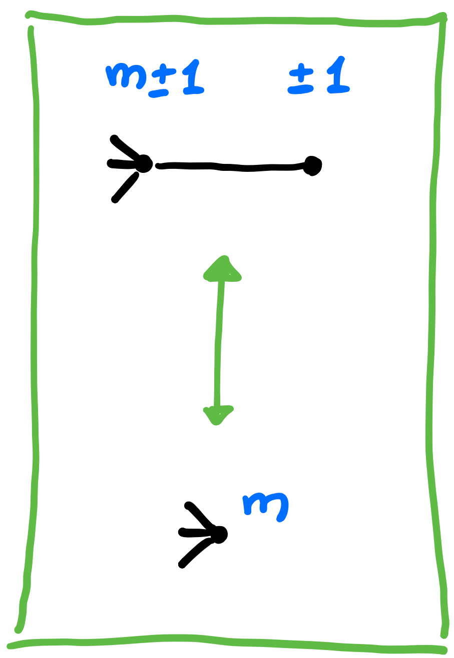
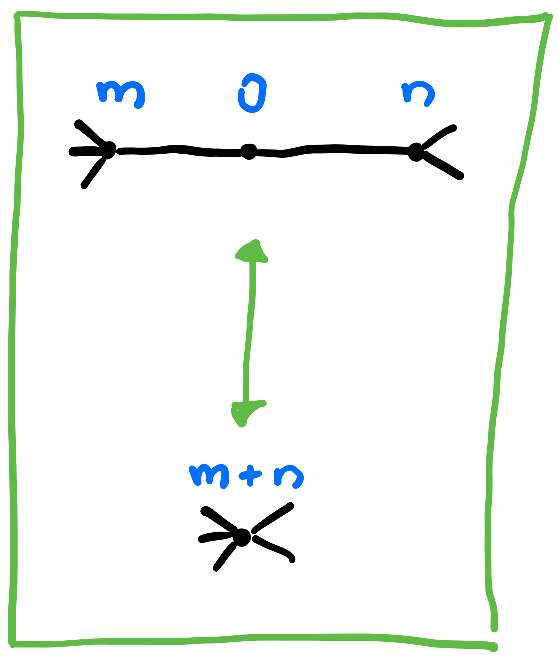
Note The only 5 moves between plumbing trees are:



(A±)



(B±)



(C)

The series $\hat{Z}$

[GPPV, 2020] For a plumbing tree Γ,

$$\hat{Z}_a(q) := (-1)^{\Delta} q^{\square} \sum_{\ell \in \mathbb{Z}^{V(\Gamma)}} q^{-\frac{1}{4}\ell^t B^{-1} \ell} \prod_{v \in V(\Gamma)} \boxed{\text{v.p.} \oint_{|z_v|=1} \left(z_v - z_v^{-1}\right)^{2-\deg v} z_v^{-\ell_v} \frac{dz_v}{2\pi i z_v}}$$

$\ell \in a + 2B\mathbb{Z}^{V(\Gamma)}$

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Note For $\deg v \geq 3$, there exists no unique inverse $\left(z - z^{-1}\right)^{-1}$:

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* for $|z| < 1$, the series exp. at 0 is: $-\sum_{i \geq 0} z^{2i+1} =: P_+(z)$
* for $|z| > 1$, —— " —— ∞ : $\sum_{i \geq 0} z^{-2i-1} =: P_-(z)$

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Thus $$\Box = \frac{1}{2}\left[P_+^{\deg v - 2}(z_v) + P_-^{\deg v - 2}(z_v)\right]_{\ell_v}$$

Gukov - Manolescu

2021

When it exists, $\hat{Z}_a$ is invariant under the 5 Neumann moves amongst trees Γ.

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* Akhmechet - Johnson - Krushkal, 2023
* Park, 2020
* Ri, 2023

The q-series

For $\tau = \left(Q, a, \xi \right)$.

root lattice — Q; Spin^c-str. — a; $\in W^{\vee(\Gamma)}$ — ξ

$$\boxed{Y_\tau(q) := (-1)^{\Delta} q^{\square} \sum_{\substack{\ell \in Q^{\vee(\Gamma)} \\ \ell \in a + 2BQ^{\vee(\Gamma)}}} c_{\Gamma,\xi}(\ell)\, q^{-\frac{1}{8}\ell^t B^{-1} \ell}}$$

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where
$$c_{\Gamma,\xi}(\ell) = \prod_{v \in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v) \right]_{\ell_v}$$

for some fcts $P^w_n(z) = \sum_{\alpha \in Q} c_\alpha z^\alpha$ with $w \in W$, $n \geq 0$.

Thm (Moore - T., 2025) When it exists,

the series $Y_{\sim}(q)$ is invariant

under the 5 moves amongst reduced Γ

Weyl vector

$$\iff \begin{cases} ① \quad \left\{ P_n^{w}(z) \right\}_{w\in W} = \left\{ \left((-1)^{\ell(w)} \sum_{\alpha \in Q} \underline{\underline{\mathfrak{K}(\alpha)}}\, z^{-w(2\rho + 2\alpha)} \right)^{n-2} \right\} \\ ② \quad \xi \in \Xi \subset W^{v(\Gamma)} \end{cases}$$

Kostant partition fct

$\boxed{\text{set of coordinated Weyl assignments}}$

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$$\Longleftrightarrow \begin{cases} ① \quad \{P_n^w(z)\}_{w\in W} = \left\{ \left((-1)^{\ell(w)} \sum_{\alpha\in Q} \underbrace{k(\alpha)}_{\text{Kostant partition fct}} z^{-w(2\overset{\text{Weyl vector}}{\rho}+2\alpha)} \right)^{n-2} \right\} \\[2ex] ② \quad \xi \in \boxed{\Xi} \subset W^{V(\Gamma)} \end{cases}$$

set of coordinated Weyl assignments

In this case

$$\hat{Z}_{Q,a}(q) = \frac{1}{|\Xi|} \sum_{\xi\in\Xi} Y_{\tau=(Q,a,\xi)}(q)$$

The (q,t)-series

For $\tau = \big(\underset{\substack{\text{root}\\ \text{lattice}}}{Q}, \underset{\substack{\text{Spin}^c\\ \text{-str.}}}{a}, \underset{\in\, \Xi}{\xi}\big).$

$$\boxed{Y_\tau(q,t) := (-1)^{\Delta} q^{\Box} \sum_{\substack{\ell \in Q^{\vee(r)} \\ \\ \ell \in a + 2BQ^{\vee(r)}}} c_{r,\xi}(\ell)\, t^{\xi^{-1}(\ell)}\, q^{-\frac{1}{8}\ell^t B^{-1} \ell}}$$

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with
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Thm (MT, 2025)

① $Y_\tau(q,t)$ exists for all reduced Γ.

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Rmk If $Y_\tau(q,t)$ can be evaluated at $t=1$, then $Y_\tau(q,1) = Y_\tau(q)$

Plumbed Knot Complements

$$(\Gamma, v_0) \longmapsto M(\Gamma, v_0) \setminus \text{tubular neighborhood of knot corr. to } v_0$$

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For $\tau = (Q, a, \xi)$,

$$Y_\tau(q, t, z) := (-1)^\Delta q^\square \sum_{\substack{\ell \in Q^{V(\Gamma)} \\ \ell \in a + 2BQ^{V(\Gamma)}}} c_{\Gamma, \xi, v_0}(\ell)\, t^{\xi^{-1}(\ell)}\, q^{-\frac{1}{8}\ell^t B^{-1}\ell}$$

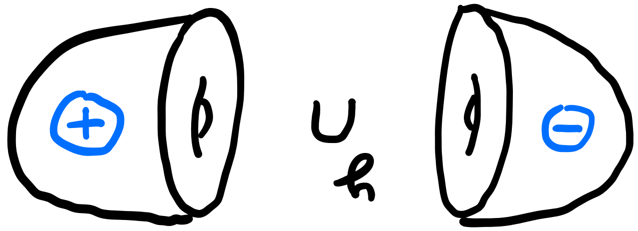
where

$$c_{\Gamma, \xi, v_0}(\ell) := z^{-\ell_{v_0}}\, P^{\xi(v_0)}_{1+\deg v_0}(z) \prod_{v \neq v_0} \left[P^{\xi(v)}_{\deg v}(z_v) \right]_{\ell_v}$$

A Gluing Formula

plumbed 3-mfd

plumbed knot complements

Thm (MT, 2025) For $M =$  .

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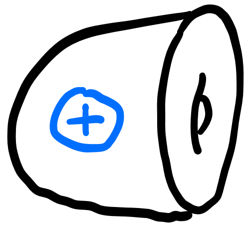
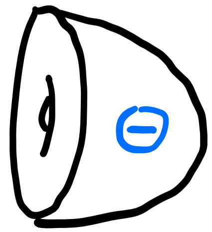
Thm (MT, 2025) For $M =$  $\cup_h$  .

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This extends and refines the gluing formula from [Gukov-Manolescu] for $Q = A_1$ and $\hat{Z}(q)$

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Expected properties [GPPV]:

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2. $\hat{Z}(q)$ is a quantum modular form, à la Zagier

→ Ex For the Poincaré hom. sphere,

$\hat{Z}(q)$ recovers the Lawrence-Zagier series (1999)

the prototypical example of a q. m. form

→ see [Liles-McSpirit], [Liles] for Seifert mfds

Dreams of a 3D TQFT

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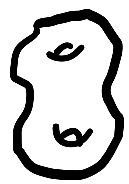
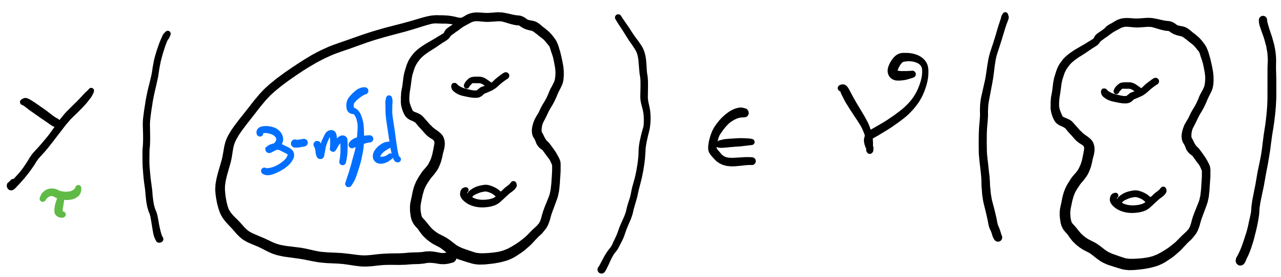
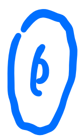
* Construct v. sp. $\mathcal{V}\Big($  $\Big)$ / Novikov-type field + gluing conditions...

s.t.



* For $g=1$, $\mathcal{V}\Big($  $\Big)$ given by Gukov-Manolescu

Thank you

for your attention!